# Cost, q-normality, and inneramenable equiv. relations

<u>Def</u> A graphing $\mathcal{G}$ of an eq rel $R$ on $(X,\mu)$ is a msble graph w/ vertex set $X$ st the connectedness relation $R_{\mathcal{G}} = R$

<u>Def</u> The cost of msble $A \subseteq R$ is

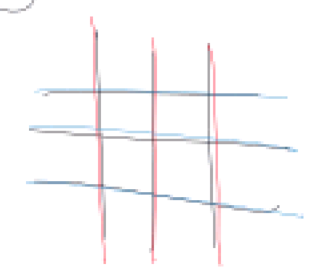
$$C_\mu(A) = \frac{1}{2}\int_X |A_x| \, d\mu(x)$$

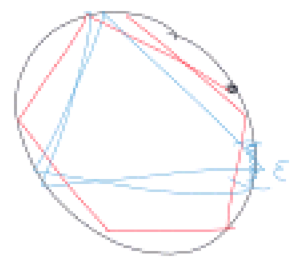
$\uparrow$ $\Sigma(p_1^{-1}(x))$ counting measure

<u>Def</u> The cost of $R$ is

$$C_\mu(R) = \inf_{\mathcal{G} \text{ graphing of } R} C_\mu(R)$$

<u>Eg</u> $\mathbb{Z}^2 \curvearrowright \mathbb{T}^1$ by mutually irrational rotations






so $C_\mu(\mathbb{Z}^2 \curvearrowright \mathbb{T}^1) = 1$

<u>Eg.</u>

| Groups | Cost |
| --- | --- |
| • infinite amenable $\Gamma \curvearrowright X$ free p.m.p. (Ornstein-Weiss) | 1 |
| • finite gps $\Gamma$, all $\Gamma \curvearrowright X$ free p.m.p. | $1 - 1/\lvert\Gamma\rvert$ |
| • inner amenable gps (Tucker-Drob) | 1 |
| • property (T) gps $\Gamma$, $\exists\, \Gamma \curvearrowright (X,\mu)$ (Hutchcroft-Pete) | 1 |
| • $\mathbb{F}_n$ (Gaboriau | $n$ |

Thm (Kechris)

If $R$ has a asymptotically central nontrivial sequence $(T_n) \subseteq [R]$ then $C_\mu(R) = 1$

$\mu(\{x \mid T_n T x = T T_n x\}) \to 1$ $\forall T \in [R]$

$\mu(\{x \mid T_n x = x\}) \not\to 1$

Prop: If $\Lambda \triangleleft_q \Gamma$, $\mathrm{Cost}(\Gamma) \leq \mathrm{Cost}(\Lambda)$

if $[\Gamma : \Lambda]$ finite (infinite) $\to [\Gamma : \Lambda](\mathrm{Cost}(\Gamma) - 1) = \mathrm{Cost}(\Lambda) - 1$

Def An inclusion of gps $\Lambda \leq \Gamma$ is called q-normal, written $\Lambda \triangleleft_q \Gamma$

if $\langle \{\gamma \in \Gamma \mid \Lambda \cap \gamma \Lambda \gamma^{-1} \text{ infinite}\} \rangle = \Gamma$

Recall Lemma If $\Gamma$ is nonamenable and $\Gamma \curvearrowright X$ is amenable w/ mean $m$.

then $m(\{x \mid \Gamma_x \text{ is nonamenable}\}) = 1$

Prop If $\Lambda \leq \Gamma$ $\Lambda$ nonamenable, $\Gamma$ inner amenable w/ mean $m$.

then $m(\{\gamma \in \Gamma \mid \Lambda \cap C_\Gamma(\gamma) \text{ is nonamenable}\}) = 1$

Pf: $\Lambda \curvearrowright \Gamma$ by conjugation

Thm

If $\Lambda \leq \Gamma$ $\Lambda$ nonamenable, $\Gamma$ inner amenable

Then $\exists K = \langle \{\gamma \in \Gamma \mid \gamma \Lambda \gamma^{-1} \cap \Lambda \text{ is nonamenable}\} \rangle$

$\Lambda \triangleleft_q K \triangleleft_q \Gamma$

Pf $\Lambda \triangleleft_q K$ immediate

$\Lambda \cap C_\Gamma(\gamma) \subseteq \Lambda \cap \gamma \Lambda \gamma^{-1}$

if $\Lambda \cap C_\Gamma(\gamma)$ nonamenable

then $\Lambda \cap \gamma \Lambda \gamma^{-1}$ is nonamenable

from above $m(\{\gamma \in \Gamma \mid \Lambda \cap \gamma \Lambda \gamma^{-1} \text{ nonamenable}\}) = 1$

$m(K) = 1$

$\forall \gamma \in \Gamma$ $m(K \cap \gamma K \gamma^{-1}) = 1$

$\Rightarrow K \triangleleft_q \Gamma$

Prop If $\Gamma$ is nonamenable

$\exists$ f.g. subgp of cost at most 2

Consequence, if $\Gamma$ is inner amenable,

$C(\Gamma) \leq 2$

Sa a groupoid $\mathcal{G}$ with unit space $(\mathcal{G}^0, \mu^0)$

has maps $r: \mathcal{G} \to \mathcal{G}^0$ $s: \mathcal{G} \to \mathcal{G}^0$

and comp. / inverse maps

$\mu^1 = \mu^0 \times C$

$= C \times \mu^0$